Mon. Not. R. Astron. Soc. **000**, 000–000 (1997) Printed February 1, 2008 (MN LATEX style file v1.4)

Gravitational lensing and the angular-diameter distance relation

Fedja Hadrović and James Binney
Theoretical Physics, 1 Keble Road, Oxford OX1 3NP

February 1, 2008

ABSTRACT
We show that the usual relation between redshift and angular-diameter distance can be derived by considering light from a source to be gravitationally lensed by material that lies in the telescope beam as it passes from source to observer through an otherwise empty universe. This derivation yields an equation for the dependence of angular-diameter on redshift in an inhomogeneous universe. We use this equation to model the distribution of angular-diameter distance for redshift $z = 3$ in a realistically clustered cosmology. This distribution is such that attempts to determine q_0 from angular-diameter distances will systematically underestimate q_0 by ~ 0.15, and large samples would be required to beat down the intrinsic dispersion in measured values of q_0.

Key words: Cosmology – Gravitational lensing

1 INTRODUCTION

The large-scale structure of the Universe is believed to be closely approximated by one of the Friedmann–Lemaître cosmological models. These are characterized by the values of three parameters: the matter density, the radius of curvature of spatial sections and the cosmological constant. Determination of these values has long been considered one of the fundamental tasks of observational cosmology. In this connection a potentially key observable is the relationship between redshift z and the angular-diameter distance $D(z)$, which is defined to be the ratio of the linear diameter of an object to the angular diameter that it subtends when observed at redshift z. There have been may attempts to determine $D(z)$ (Sandage 1988; Kellermann 1993; Crawford 1995) which has recently prompted a wider discussion about the feasibility of the method (Nilsson et al. 1993; Dabroski, Lasenby & Saunders 1995; Kantowski, Vaughan & Branch 1995; Stephanas & Saha 1995). The form of $D(z)$ depends on the geometry of the Universe in the sense that the larger the curvature K is, the smaller D is at a given redshift. That is, the more positively curved the Universe is, the more slowly the angular size of an object decreases as it is moved away from the observer.

In an inhomogeneous universe deviations of the metric from the Friedmann–Lemaître form give rise to fluctuations in measured values of D at fixed z. Previous analyses of this effect (Zel'dovich, 1964; Refsdal, 1970; Dyer & Roeder, 1974; Sasaki 1987; 1; Watanabe & Tomita 1990) worked directly from general relativity and produced results of considerable complexity. By delegating relativistic considerations to the theory of gravitational lensing (e.g. Schneider, Ehlers & Falco, 1992), we obtain a much simpler analysis and an equation (15) that involves the cosmic density field rather than the cosmic metric or potential. This simplicity enables us to evaluate the effects of inhomogeneity for the case of realistic clustering, rather than the case of either weak perturbations (Sasaki 1987) or randomly distributed point masses (Zel'dovich, 1964; Refsdal, 1970; Dyer & Roeder, 1974; Watanabe & Tomita 1990; Kantowski, Vaughan & Branch 1995).

Over the last decade there has been a growing awareness of the importance of gravitational lensing for observations of high-redshift objects. Gravitational lensing and the dependence of D on z are two sides of the same coin: both phenomena are caused by the tendency of matter that lies between the observer and a distant object to focus radiation from that object, thereby increasing its apparent size and brightness. In Section 2 we demonstrate this connection quantitatively by showing that the standard formula for $D(z)$ in a Friedmann–Lemaître universe can be obtained by applying conventional lensing theory to a Universe in which there is matter *only* in the telescope beam towards the object under study. In Section 3 we describe our model of the clustered cosmic density field, and in Section 4 we use this model to calculate probability distributions for $D(z)$ from objects of various linear sizes. Section 5 sums up.

We throughout use the convention that $\widetilde{D}$ and D, respectively, denote angular-diameter distance before and after lensing is taken into account.

Since luminosity distance D_L is rigorously related to angular-diameter distance by $D_L = D/(1+z)^2$ (Etherington 1993), our distributions of values of D imply identical distributions of values of D_L.

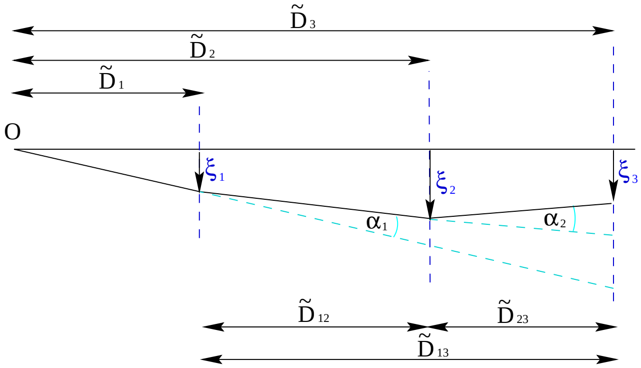

Figure 1. A sequence of gravitational lenses

The unit system we use is based on $G = c = H_0 = 1$, which significantly simplifies the equations in cosmology. All lengths quoted are scaled to $H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

2 THE ANGULAR-DIAMETER DISTANCE RELATION FROM LENSING

2.1 A sequence of gravitational lenses

Fig. 1 shows a ray that is deflected through angles $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ by two lenses, which it passes at impact parameters $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, respectively. From the figure it is immediately apparent that

$$\boldsymbol{\xi}_3 = \frac{\widetilde{D}_3}{\widetilde{D}_1}\boldsymbol{\xi}_1 - \boldsymbol{\alpha}(\boldsymbol{\xi}_1)\widetilde{D}_{13} - \boldsymbol{\alpha}(\boldsymbol{\xi}_2)\widetilde{D}_{23}. \tag{1}$$

The generalization of this equation to an arbitrary number of lenses is easily seen to be

$$\boldsymbol{\xi}_j = \frac{\widetilde{D}_j}{\widetilde{D}_1}\boldsymbol{\xi}_1 - \sum_{i=1}^{j-1}\widetilde{D}_{ij}\boldsymbol{\alpha}_i(\boldsymbol{\xi}_i). \tag{2}$$

Light that passes at radius vector $\boldsymbol{\xi}$ through a disc of matter that has uniform surface density Σ is deflected through an angle

$$\boldsymbol{\alpha} = 4\pi\Sigma\boldsymbol{\xi}. \tag{3}$$

Hence in this case the impact parameters are all parallel and satisfy

$$\boldsymbol{\xi}_j = \frac{\widetilde{D}_j}{\widetilde{D}_1}\boldsymbol{\xi}_1 - 4\pi\sum_{i=1}^{j-1}\Sigma_i\widetilde{D}_{ij}\boldsymbol{\xi}_i. \tag{4}$$

Finally, if $\boldsymbol{\xi}_i$ is a diameter of an object, then the angular-diameter distance of that object is $D_i \equiv \xi_i/(\xi_1/\widetilde{D}_1)$, so on taking the modulus of (4) and dividing through by $(\xi_1/\widetilde{D}_1)$ we conclude that true angular-diameter distances D_i satisfy

$$D_j = \widetilde{D}_j - 4\pi\sum_{i=1}^{j-1}\Sigma_i D_i\widetilde{D}_{ij} \tag{5}$$

2.2 Application to a homogeneous universe

We now we show that equation (5) reproduces the familiar angular-diameter distance equation for a Friedmann–Lemaître universe. We consider an empty universe. In such a universe there is nothing to single out a unique rest frame at any given event, so redshift is not uniquely related to distance. This permits us simply to adopt the relation $s(z)$ between proper distance and redshift in a Friedmann–Lemaître universe. We have that $s(z)$ satisfies [Scheneider et al., eq. (4.47b)]

$$\frac{\mathrm{d}s}{\mathrm{d}z} = (1+z)^{-2}(1+\Omega z)^{-1/2}. \tag{6}$$

From the gravitational-focusing equation [Schneider et al. eq. (3.64)] in the case of empty space (vanishing Ricci tensor and shear) we have

$$\frac{\mathrm{d}^2\widetilde{D}}{\mathrm{d}\tau^2} = 0, \tag{7}$$

where τ is an affine parameter for the light beam. In terms of the wavenumber, k, we have $\mathrm{d}s/\mathrm{d}\tau \propto k \propto 1+z$, so we may use equation (6) to eliminate τ from (7) in favour of s. We then find that the focusing equation states that in our empty universe, as a function of z, angular-diameter distance $\widetilde{D}$ satisfies

$$(1+z)(1+\Omega z)\ \frac{\partial^2\widetilde{D}(y,z)}{\partial z^2} + \left(\tfrac{7}{2}\Omega z + \tfrac{1}{2}\Omega + 3\right)\frac{\partial\widetilde{D}(y,z)}{\partial z} = 0. \tag{8}$$

We also have the initial condition [Schneider et al. eq. (4.53)]

$$\left.\frac{\partial\widetilde{D}(y,z)}{\partial z}\right|_{y=z} = (1+z)^{-2}(1+\Omega z)^{-1/2}. \tag{9}$$

Now we fill the telescope beam with the normal matter density of a Friedmann–Lemaître universe and use equation (5) to calculate the angular-diameter distance of an object at 'redshift' z. We first take the limit of equation (5) in which there are an infinite number of discs. Since in our units the current critical density is $3/(8\pi)$, the disc that lies between $z+\mathrm{d}z$ and z has surface density

$$\Sigma = (1+z)^3\frac{3\Omega}{8\pi}\frac{\mathrm{d}s}{\mathrm{d}z}\mathrm{d}z, \tag{10}$$

where Ω is the usual density parameter. With equation (6) this becomes

$$\Sigma(z) = \frac{3\Omega}{8\pi}\frac{1+z}{\sqrt{1+\Omega z}}\mathrm{d}z. \tag{11}$$

Inserting this expression for Σ into equation (5) and proceeding to the limit $\mathrm{d}z \to 0$ we find

$$D(z) = \widetilde{D}(z) - \tfrac{3}{2}\Omega\int_0^z \mathrm{d}y\frac{1+y}{\sqrt{1+\Omega y}}D(y)\widetilde{D}(y,z). \tag{12}$$

We now convert this integral equation for $D(z)$ into a differential equation. Differentiating we find

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}z} &= \frac{\mathrm{d}\widetilde{D}}{\mathrm{d}z} - \tfrac{3}{2}\Omega\int_0^z \mathrm{d}y\,\frac{1+y}{\sqrt{1+\Omega y}}D(y)\left(\frac{\partial}{\partial z}\widetilde{D}(y,z)\right),\\ \frac{\mathrm{d}^2D}{\mathrm{d}z^2} &= \frac{\mathrm{d}^2\widetilde{D}}{\mathrm{d}z^2} - \tfrac{3}{2}\Omega\,\frac{1+z}{\sqrt{1+\Omega z}}D(z)\left(\frac{\partial}{\partial z}\widetilde{D}(y,z)\right)_{y=z}\\ &\quad -\tfrac{3}{2}\Omega\int_0^z \mathrm{d}y\,\frac{1+y}{\sqrt{1+\Omega y}}D(y)\left(\frac{\partial^2}{\partial z^2}\widetilde{D}(y,z)\right).\end{aligned} \tag{13}$$

Combining these equations and taking advantage equations (8) and (9), we recover the standard equation for

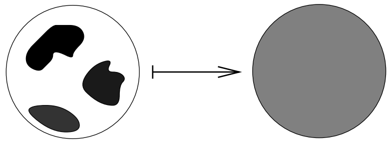

Figure 2. Averaging of the matter distribution over the beam cross section

the angular-diameter distance in a conventional Friedmann–Lemaître universe:

$$(1+z)(1+\Omega z)\frac{\partial^2 D}{\partial z^2}+\left(\tfrac{7}{2}\Omega z+\tfrac{1}{2}\Omega+3\right)\frac{\partial D}{\partial z}+\tfrac{3}{2}\Omega D=0. \quad (14)$$

2.3 Application to an inhomogeneous universe

The most important feature of the above derivation is that it does not depend on Ω being constant. In the first two terms of equation (14) Ω appears as a result of the reparametrisation ($s \mapsto z$). It is not related to the local matter distribution and can be thought of as the averaged density parameter $\langle\Omega\rangle$. The parameter Ω in the last term, $\frac{3}{2}\Omega D$ *is* related to the local matter density and comes directly from the gravitational lensing calculation. Therefore, in the case of a locally inhomogeneous universe that approaches a Friedmann–Lemaître model in the large-scale limit, we can write

$$(1+z)(1+\langle\Omega\rangle z)\frac{\partial^2 D}{\partial z^2}+\left(\tfrac{7}{2}\langle\Omega\rangle z+\tfrac{1}{2}\langle\Omega\rangle+3\right)\frac{\partial D}{\partial z} +\tfrac{3}{2}\Omega(z)D=0. \quad (15)$$

Note that $\Omega(z)$ describes the comoving matter density because the physical density is $\rho(z)=\frac{3}{8\pi}\Omega(1+z)^3$.

Simply replacing Ω by $\langle\Omega\rangle$ in all but the last term of equation (14) does not allow for a complete discussion of the effects of inhomogeneity on images: in addition to being magnified by matter within the beam, images will be distorted and may be even split into multiple images. We have neglected these potentially important effects by (i) assuming that the material that lies between redshifts $z+\mathrm{d}z$ and z forms a uniform disc, and (ii) neglecting shear that is induced by clumps of material that lie outside the beam. Futumase & Sasaki (1989) and Watanabe & Sasaki (1990) show that as long as the scale of inhomogeneities is greater than, or equal to galactic scale, shear does not contribute significantly to focusing.

By contrast, the assumption that the beam is filled by a series of uniform-density discs constitutes a non-trivial approximation about the matter distribution in the beam, namely that we may average the density across the beam as shown in Fig. 2.

3 STATISTICAL MODEL OF THE FIELD $\Omega(\boldsymbol{r})$

We now investigate the predictions of the generalized diameter-distance equation (15). For this investigation we require a statistical description of the density field along the telescope beam. This is a random field, which we think of as a function of comoving distance x. We assume that $\Omega(x)$ follows a log-normal distribution – see Coles & Jones (1991) for a discussion of the characteristics and advantages of the log-normal distribution in cosmology. We confine ourselves to the case of a critical-density universe: $\langle\Omega\rangle=1$. With these assumptions $\Omega(x)$ is given by

$$\Omega(x)=\frac{\mathrm{e}^{\varepsilon(x)}}{\langle\mathrm{e}^{\varepsilon}\rangle}, \quad (16)$$

where $\varepsilon(x)$ is a Gaussian random field. Without loss of generality we set $\langle\varepsilon\rangle=0$.

We define the two-point correlation function, ξ_f of a field $f(x)$ by

$$\xi_f(x)=\frac{\langle f(x'+x)f(x')\rangle-\langle f(x')\rangle^2}{\langle f(x')^2\rangle-\langle f(x')\rangle^2}. \quad (17)$$

The correlation functions of the fields $\Omega(x)$ and $\varepsilon(x)$ are related by

$$\xi_\Omega(x)=\frac{\exp(\sigma_\varepsilon^2\xi_\varepsilon(x))-1}{\exp(\sigma_\varepsilon^2)-1}, \quad (18)$$

where $\sigma_\varepsilon^2=\left\langle\varepsilon^2\right\rangle$ is the variance of the Gaussian field.

The Gaussian field $\varepsilon(x)$ is determined by its power spectrum $P_\varepsilon(k)$, which is essentially the Fourier transform of ξ_ε:

$$P_\varepsilon(k)=\frac{\sigma_\varepsilon^2}{2\pi}\int \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}\xi_\varepsilon(x). \quad (19)$$

Hence, if we know $\xi_\Omega(x)$, we may construct realizations of Ω by determining $\xi_\varepsilon(x)$ from equation (18) and then using equation (19) to determine $P_\varepsilon(k)$.

The galaxy correlation function may be approximated by (Padmanabhan, 1993)

$$\xi(r)=\left(\frac{r}{r_c}\right)^{-\gamma}, \quad (20)$$

where $\gamma\approx 1.8$ and the correlation length is $r_c\simeq 5.5h^{-1}$Mpc. The correlation function of the density field is often assumed to have the same form, but a different amplitude. The bias factor is introduced by setting

$$b=\frac{\xi_{\text{GALAXIES}}}{\xi_{\text{MATTER}}}. \quad (21)$$

Measurements indicate that $1\lesssim b\lesssim 2$. Hence we require ξ_Ω such that $\xi_\Omega(0)=1$ and

$$\sigma_\Omega^2\xi_\Omega(r)\approx b^{-1}\left(\frac{r}{r_c}\right)^{-\gamma}. \quad (22)$$

We have adopted the form

$$\xi_\Omega(r)=\left(1+\frac{r^2}{r_0^2}\right)^{-1} \quad (23)$$

with $\sigma_\Omega r_0 = b^{-1/2}r_c$. For $b = 1.5$ we find $\sigma_\Omega r_0 = 4.5h^{-1}$Mpc. The meaning of r_0 will be discussed later, but we immediately see that for small r_0 the model approximates the divergent galaxy correlation function better.

From equation (18) we find

$$\xi_\varepsilon(r)=\frac{1}{\sigma_\varepsilon^2}\ln\left(\frac{r_0^2\mathrm{e}^{\sigma_\varepsilon^2}+r^2}{r_0^2+r^2}\right). \quad (24)$$

From (19) the power spectrum is

$$\begin{aligned}|P_\varepsilon(k)|^2&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\ln\frac{r_0^2\mathrm{e}^{\sigma_\varepsilon^2}+x^2}{r_0^2+x^2}\mathrm{e}^{\mathrm{i}kx}\\ &=\frac{1}{k}\left[\exp(-kr_0)-\exp(-kr_0\mathrm{e}^{\sigma_\varepsilon^2/2})\right]. \quad (25)\end{aligned}$$

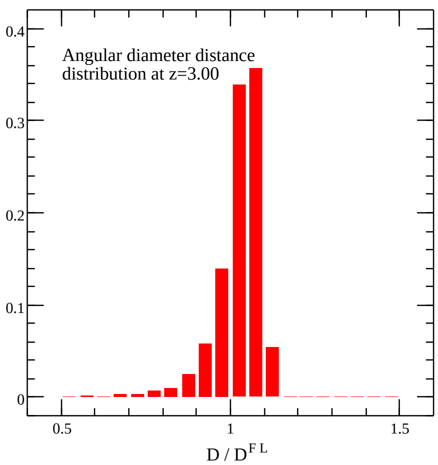

Figure 3. The distribution of angular-diameter distance at $z = 3$

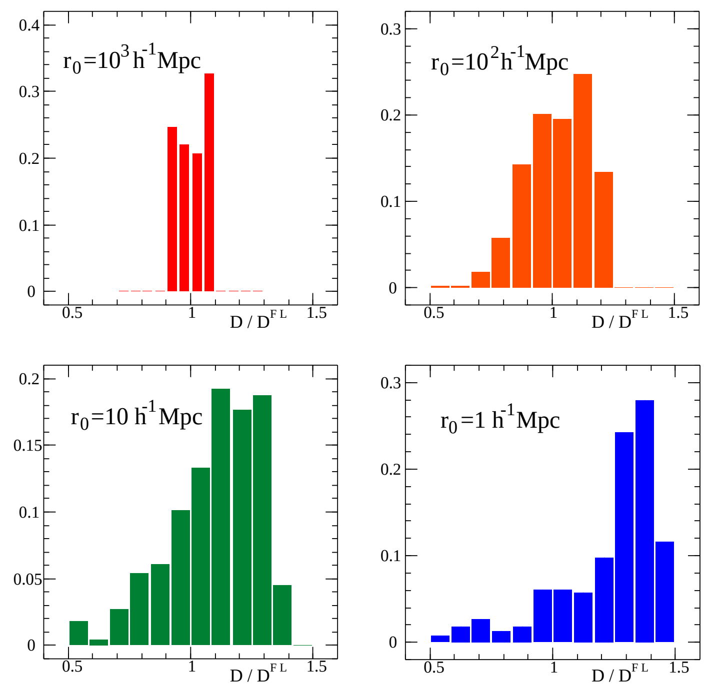

Figure 4. Effect of smoothing on the angular diameter distance distribution

The significance of r_0 now emerges: it determines how quickly $|P_\varepsilon(k)|$ approaches zero at large k. The smaller the value of r_0 the larger must be the wavenumber $k_{\max}$ up to which we must sum the discrete Fourier transform from which we obtain realizations of $\Omega(x)$. Physically, we should think of r_0 as the scale on which the matter distribution is smoothed by the finite width of our telescope beam and the diameters of the objects we are looking at. If we take $r_0 = 10h^{-1}$ kpc, we have $\sigma_\varepsilon^2 = 12.21$.

Due to computational constraints and limitations on sampling imposed by Nyquist's theorem, it was impracticable to generate a single random field on the range $0 < z < 3$. Instead, we divided this interval into 100 subintervals and create a scaled random field on each of them. This procedure destroys correlations between different intervals but these are physically unimportant because the correlation function is negligible at such large distances.

4 RESULTS AND DISCUSSION

4.1 Distribution of angular-diameter distances

Once a realization of $\Omega(x)$ has been constructed, it is straightforward to solve equation (15) for D at any given value of z. We repeated this operation for approximately 4000 realizations of $\Omega(x)$ to determine the distribution of angular-diameter distances at $z = 3$. Fig. 3 shows this distribution. The distances are rescaled to the standard Friedmann–Lemaître value

$$D^{\rm FL} = \frac{2}{1+z}\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{26}$$

An important point on the graph is the Dyer–Roeder distance corresponding to an empty light beam:

$$D^{\rm DR} = \tfrac{2}{5}\left(1 - \frac{1}{(1+z)^{5/2}}\right), \tag{27}$$

which for $z = 3$ gives $D^{\rm DR}/D^{\rm FL} = 1.55$.

We see that the distribution is strongly peaked on the Dyer–Roeder side of $D^{\rm FL}$, with a long tail on the Friedmann–Lemaître side. This is expected because regions within which the density is below average occupy the great majority of the volume of the Universe. Hence, many light paths sample only low-density regions and the distribution in Fig. 3 is shifted towards $D^{\rm DR}$. However, when the light beam does encounter a galaxy or other matter aggregation, it is strongly lensed. These events decrease the diameter distance and give rise to the tail on the Friedmann–Lemaître side.

It is important to understand the impact that smoothing of the matter distribution has on our results. It is computationally convenient to investigate this for an unrealistic case: we take the correlation length to be 100 times its true value. That is, we investigate the case in which $\sigma_\Omega r_0 = 450h^{-1}$Mpc. Fig. 4 shows our results.

For large r_0 the matter distribution is rather homogeneous, so the distribution of D is narrow and peaked near $D^{\rm FL}$. As r_0 is decreased the universe becomes strongly inhomogeneous and the distribution of D becomes broader. Simultaneously, its peak shifts towards the empty-beam distance $D^{\rm DR}$.

4.2 Implications for q_0 measurements

One of the most important undetermined quantities of cosmology is q_0, the deceleration parameter. For a flat universe ($K = 0$), the angular-diameter distance D is related to q_0 by

$$D(q_0, z) = \frac{R}{1+z}, \tag{28}$$

where

$$R \equiv \int_1^{1+z} \frac{{\rm d}u}{\left(\Omega u^3 + (1 + q_0 - 3\Omega/2)u^2 + \frac{1}{2}\Omega - q_0\right)^{1/2}}. \tag{29}$$

Suppose we attempt to use (28) to determine q_0 from an observationally determined value of $D(z)$. We assume that the true values of the cosmic constants are those with which we have been working: $\Omega = 1$, $\Lambda = K = 0$, and thus that the true value of q_0 is $q_0 = \frac{1}{2}$. Putting $q_0 = \frac{1}{2} + \delta q_0$ in equation (29) we find

$$R(\delta q_0) = \int_1^{1+z} \frac{\mathrm{d}u}{u^{3/2}} \left(1 - \delta q_0 \frac{u^2 - 1}{2u^3}\right). \tag{30}$$

Substituting this value of R into (28) gives

$$\frac{D}{D^{\mathrm{FL}}}\bigg|_{z=3} = 1 - 0.15\, \delta q_0. \tag{31}$$

This equation relates the error, δq_0, in the inferred value of q_0 to the ratio of the measured value of D to the value D^{FL} that it would have if the Universe were homogeneous. The distribution of D/D^{FL} shown in Fig. 3 is centred on 1.025 and has spread $\sim \pm 0.06$. By equation (31) the error in q_0 to which this gives rise is

$$\delta q_0 = -0.17 \pm 0.4. \tag{32}$$

In connection with this result three points should be made:

- We see that the conventional method of determining q_0 from the angular-diameter redshift relation provides a biased estimator of q_0 that will return significant underestimates of the true value.
- Even perfect measurements of $D(z)$ will return values of q_0 that are widely scattered. The breadth of this scatter is such that an accurate determination of q_0 would require an extremely large sample and a sophisticated statistical analysis of the data.
- The errors in q_0 to which inhomogeneities give rise depend on the scale of observed objects because this scale determines the effective spectrum of the inhomogeneities. Larger objects will yield smaller errors.

This last point is unfortunate because, as Kellermann (1993) has emphasized, small objects are much more likely to constitute standard measuring rods than large objects, such as giant radio sources, whose linear sizes are likely to be sensitive to the mean cosmic density.

5 CONCLUSION

We have used the theory of gravitational lensing to derive the conventional relation between angular-diameter distance and redshift in a Friedmann–Lemaître universe. The value of this derivation is that it is simple and shows that the tendency of the angular diameter of a distant object to increase with Ω arises because rays coming from the object are focused by matter that lies within the telescope beam. Hence, the angular diameter of an object is sensitive to the precise disposition of matter in the neighborhood of the telescope beam: move matter just out of the beam and the apparent size of the object will diminish. Equation (15) expresses this fact mathematically.

Since the Universe is strongly inhomogeneous on small scales, telescope beams to different objects at the same redshift will contain significantly varying quantities of matter, and the apparent diameters of physically identical objects at a common redshift will vary. This variation gives rise to scatter in the angular-diameter distances D of a set of objects that lie at a common redshift.

We have modelled the distribution of the values of D of objects at redshift $z = 3$ by assuming that the cosmic density field follows a lognormal distribution that matches the observed clustering of galaxies for bias parameter $b = 1.5$. The distribution of D is very skew, with its peak at a value that exceeds that associated with the corresponding homogeneous universe, D^{FL}, and a long tail to values smaller than D^{FL}. In consequence of this skewness, the conventional technique for measuring q_0 from measurements of $D(z)$ will systematically underestimate q_0.

The width of the distribution of D at given z depends upon the assumed power spectrum $P(k)$ of the cosmic density field. The true power spectrum is thought to have considerable power on small scales, and this power will generate a very broad distribution of D for objects of small angular size. When the angular diameters of highly extended objects are measured, only power on scales comparable to or larger than the linear size r_0 of the objects will contribute to the scatter in D. Hence such measurements will yield less scattered values of D. For $r_0 = 10h^{-1}$kpc we estimate that D will scatter by $\sim \pm 6\%$ at redshift $z = 3$. Unfortunately, even this small scatter will cause the derived values of q_0 to scatter by as much as ± 0.4. The scatter in values of q_0 that are derived from angular-diameter distances to parsec-sized objects such as those studied by Kellermann (1993), will be very much larger still.

REFERENCES

Coles P., Jones B., 1991, MNRAS 248, 1
Crawford D.F., 1995, ApJ 440, 466
Dabrowski Y., Lasenby A., Saunders R., 1995, MNRAS, 277, 753
Dyer C.C., Roeder R.C., 1974, ApJ, 189, 167
Etherington I.M.H., 1933, Phil. Mag., 15, 761
Futamase T., Sasaki, M., 1989, Phys. Rev. D, 40, 2502
Kantowski R., Vaughan T., Branch D., 1995, ApJ 447, 35
Kasai M., Futamase T., Takahara F., 1990, Phys. Lett. A, 147, 97
Kellermann K.I., 1993, Nature, 361, 134
Nilsson K., Valtonen M.J., Kotilainen J., Jaakkola T., 1993, ApJ 413, 453
Padmanabhan T., 1993, 'Structure Formation in the Universe,' Cambridge University Press, Cambridge
Refsdal S., 1970, ApJ, 159 357
Sandage A., ARA&A, 26, 561
Sasaki M., 1987, MNRAS, 228, 653
Schneider P., Ehlers J., Falco E. E. 1992, 'Gravitational Lenses,' Springer Verlag, Berlin
Stephanas P.G., Saha P., 1995 MNRAS 272, L13
Watanabe K., Sasaki M., 1990, PASJ 42, L33
Watanabe K., Tomita K., 1990, ApJ 355, 1
Zel'dovich Ya.B., 1964, Sov. Astron., 8, 13